15.4 Lecture: Polar integrals and Polar area

Jeremiah Southwick
(But really Robert Vandermolen)

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Links

Robert's slides can be found here:

http://people.math.sc.edu/robertv/teaching.html

The 15.4 slides can be found here:

https://docs.google.com/presentation/d/
1-xUgXkCZh0CglmUvGODH8yx6bjv6z4nGlkZLxVPeqZ8

Double Integrals with Polar Coordinates!

As with rectangular coordinates, we can bound by functions as well...

If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then $$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Double Integrals with Polar Coordinates!

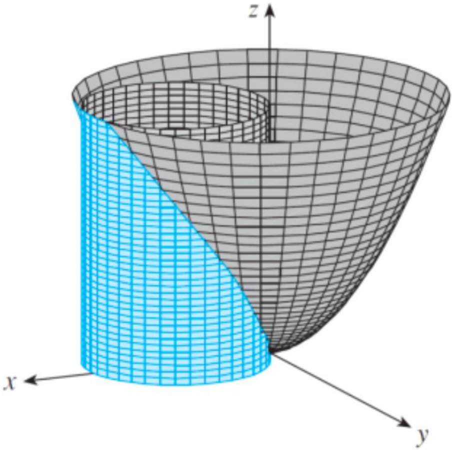
Example:

Find the volume of the solid that lies under the paraboloid,

$$z = x^2 + y^2$$

above xy-plane, and inside the cylinder,

$$x^2 + y^2 = 2x$$



Double Integrals with Polar Coordinates!

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above xy-plane, and inside the cylinder,

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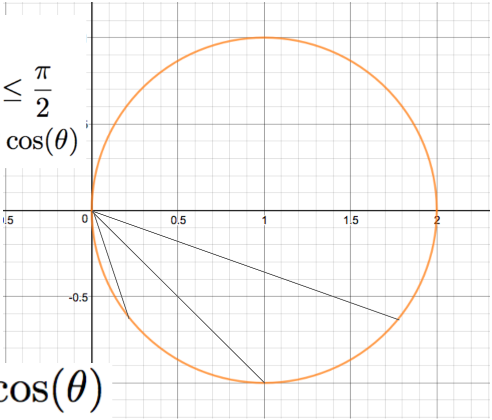
$$\Downarrow$$

$$r^2 = 2r\cos(\theta) \Rightarrow r = 2\cos(\theta)$$

$R:$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$0 \leq r \leq 2\cos(\theta)$$



Double Integrals with Polar Coordinates!

Example:

$R:$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$0 \leq r \leq 2\cos(\theta)$$

Recall: $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$

$$\int_R\int x^2 + y^2 \, dA =$$

NOW YOU TRY!

Evaluate the double integrals by switching to polar coordinates:

- $$\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{3/2} \, dydx$$
- $$\int_0^2 \int_0^{\sqrt{2x-x^2}} xy \, dydx$$

Area

Theorem

The area of a region R given by polar coordinates is

$$A = \int\int_R r \; dr \; d\theta$$

Example

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Find the area enclosed by the lemniscate $r^2 = 4\cos(2\theta)$*.*

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$$A = \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=\sqrt{4\cos(2\theta)}} r \; dr \; d\theta$$

Example

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Find the area enclosed by the lemniscate $r^2 = 4\cos(2\theta)$.

(Lemniscate = Bow tie)
First we sketch the region.

$$A = \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=\sqrt{4\cos(2\theta)}} r \; dr \; d\theta = 4 \int_{\theta=0}^{\theta=\pi/4} \left[\frac{r^2}{2} \right]_{r=0}^{r=\sqrt{4\cos(2\theta)}} d\theta$$

$$= 4 \int_{\theta=0}^{\theta=\pi/4} 2\cos(2\theta) d\theta = 4\sin(2\theta) \Big]_{\theta=0}^{\theta=\pi/4} = 4.$$